

General behaviour of Bianchi VI₀ solutions with an exponential-potential scalar field.

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Abstract

The solutions to the Einstein-Klein-Gordon equations without a cosmological constant are investigated for an exponential potential in a Bianchi VI₀ metric. There exists a two-parameter family of solutions which have a power-law inflationary behaviour when the exponent of the potential, k , satisfies $k^2 < 2$. In addition, there exists a two-parameter family of singular solutions for all k^2 values. A simple anisotropic exact solution is found to be stable when $2 < k^2$.

KEY WORDS: Bianchi models, exact solutions, asymptotic structure, power-law inflation.

Inflationary theories claim that the isotropy of the Universe can be explained by assuming an inflationary expansion in the early universe. This belief is based on the “Cosmic no-hair” theorems [1]. Heusler [2], however, proved that by considering the entire evolution of the scalar field coupled to the gravitational field and assuming a very large class of reasonable potentials, the evolution of the anisotropic Bianchi models was similar to that found by Collins and Hawking [3]: only those Bianchi models that have as particular solutions the FRW models isotropize.

One of the potentials which has received special attention is that of the Liouville form. It appears in the Jordan-Brans-Dicke theory and does not belong to the class of potentials analyzed by Heusler. The asymptotic behaviour of the Bianchi models with an exponential-potential scalar field has been studied recently [4] using the techniques developed in [5]. When the constant k that gives the slope of the potential is less than $\sqrt{2}$ then the space-time inflates and isotropizes. However, when $k^2 > 2$, the models that can possibly isotropize are those of Bianchi types I, V, VII or IX. The isotropization of several scalar field Bianchi models has been recently studied in [6] and [7].

In [8] the Bianchi I models with exponential potential were analyzed reducing the problem to one third order differential equation. In [9] a non-local transformation was used to linearize this equation and the general solution was found. It allows a better insight into the problem and shows a damped oscillatory behaviour which leads to an effective negative cosmological constant. Several singular solutions representing universe models that have either anisotropic or isotropic Friedmann–Robertson–Walker final stages were found. There are also solutions which avoid the initial singularity and others with a finite time span. In this paper we shall extend this analysis to the Bianchi VI₀ type, investigating the behaviour of the solutions near the singularity and its asymptotic stability in the far future.

The metric corresponding to a Bianchi VI₀ cosmological model can be written in the following form:

$$ds^2 = e^{f(t)} \left(-dt^2 + dz^2 \right) + G(t) \left(e^z dx^2 + e^{-z} dy^2 \right). \quad (1)$$

The pertinent Klein-Gordon and Einstein field equations for the metric (1) are as follows:

$$\ddot{\phi} + \frac{\dot{G}}{G} \dot{\phi} + e^f \frac{\partial V}{\partial \phi} = 0, \quad (2)$$

$$\frac{\ddot{G}}{G} = 2e^f V, \quad (3)$$

$$\frac{\ddot{G}}{G} - \frac{1}{2} \frac{\dot{G}^2}{G^2} - \dot{f} \frac{\dot{G}}{G} + \frac{1}{2} + \dot{\phi}^2 = 0. \quad (4)$$

We will consider a homogeneous self-interacting scalar field ($\phi = \phi(t)$) with an exponential potential $V = \Lambda e^{k\phi}$. As can be easily shown

$$\dot{\phi} = \frac{m}{G} - \frac{k}{2} \frac{\dot{G}}{G}, \quad (5)$$

is a first integral of the Klein-Gordon equation and m is an arbitrary integration constant. From Eqs.(3-5), we get

$$\ddot{G}^2 G - K \ddot{G} \dot{G}^2 - \ddot{G} \dot{G} G + \frac{1}{2} \ddot{G} G^2 + m^2 \ddot{G} = 0; \quad K = \frac{k^2}{4} - \frac{1}{2}, \quad (6)$$

First we will consider the special case in which $m^2 = 0$, and after that, we will pay attention to the most general case $m^2 \neq 0$.

$m^2 = 0$ case

To analyze the asymptotical behaviour of the solutions we define the new variables $h = \frac{\dot{G}}{G}$ and $d\eta = h dt = d \ln G$, so that Eq.(6) now reads

$$h^2 h'' + (1 + K)h^2 h' + Kh^3 - \frac{1}{2}(h' + h) = 0, \quad (7)$$

with $h' = \frac{dh}{d\eta}$. The simplest solution of equation (7) is $h^2 = \frac{1}{2K}$ for $K > 0$. Hence, we get

$$G = G_0 e^{\sqrt{\frac{1}{2K}} t}, \quad (8)$$

while the remaining component of the metric f and the scalar field ϕ can be obtained from Eqs.(3) and (5) respectively.

Now, we assume that the function G vanishes or diverges at $t = 0$ and its leading term (this assumption is justified in Ref.[11]), is given by

$$G(t) \simeq G_0 t^n. \quad (9)$$

Substituting this expression into Eq.(7), we can see that the last two terms are negligible compared with the first four terms. So

$$\frac{1}{t^4} [n^2 - (1 + K)n^3 + Kn^4] \simeq 0 \quad (10)$$

is the equation that determines the values of the exponent n . Inserting this value of n and Eq.(9) in Eqs.(3) and (5), we obtain the approximate scalar field and metric in the limit $t \rightarrow 0$.

For $1 < K$ which corresponds to values of $6 < k^2$, with $0 < n < 1$, we have $\Lambda < 0$ for Eq.(3), so the potential must be negative. However, it is positive for $K < 1$.

For $-\frac{1}{2} < K < 0$ which corresponds to values of $k^2 < 2$, with $-\infty < n < -2$, the metric coefficients G and e^f diverge when $t \rightarrow 0$.

The next term in the expansion of the function G , can be obtained solving Eq.(7) when the last two terms are neglected. Its general implicit solution is

$$G = \left[-\frac{C_1}{C_2} + C_3 e^{(1-K)C_2 \tau} \right]^{\frac{1}{1-K}}, \quad K \neq 1, \quad (11)$$

where $\tau = \int G^{-K} dt$.

For $-\frac{1}{2} < K < 0$ and $1 < K$ the function G diverges (vanishes) in the limit $\tau \rightarrow \pm\infty$ according to $C_2 > 0$ or $C_2 < 0$. In this case we get

$$G(t) \simeq at + b|t|^{\frac{1}{K}}, \quad (12)$$

There exists a two-parameter family of solutions that behaves as $G \simeq |t|^{\frac{1}{K}}$. Also, G vanishes at a finite time τ_0 for $K < 1$, provided that $C_1 C_2 C_3 > 0$. Its approximate expansion is

$$G \simeq at + b|t|^{2-K}. \quad (13)$$

Thus a two-parameter family of solutions behaves as $G \simeq t$.

The behaviour and stability of the solutions can be investigated by writing Eq.(7) as the equation of motion for a dissipative or antidissipative system,

$$\frac{d}{d\eta} \left[\frac{h'^2}{2} + K \frac{h^2}{2} - \frac{1}{2} \ln h \right] = - \left[-\frac{1}{2h^2} + (1 + K) \right] h'^2, \quad (14)$$

with the potential

$$\mathcal{V}(h) = K \frac{h^2}{2} - \frac{1}{2} \ln h. \quad (15)$$

Equation (14) presents local minima when $h_0^2 = \frac{1}{2K}$, for $K > 0$ (i.e. $k^2 > 2$). As the dissipative term given by the right-hand side of Eq.(14) is negative definite in the asymptotic regime, the bracket on the l.h.s. of Eq.(14) defines a Liapunov Function [12],[13],[14]. Then, the exact solution (8) is stable for $t \rightarrow \infty$ and for any initial condition. Studying the behaviour of the solutions around these equilibrium points we have:

For $K > 0$ ($k^2 > 2$) there is a two-parameter family of stable solutions that behaves as (8). When $K > \frac{1}{8}$ it can be shown that the solutions cut the h axis in the phase plane (h, \dot{h}) an infinite number of times. Therefore they

spiral in the phase plane around the constant solution h_0 . For $-\frac{1}{2} < K \leq \frac{1}{8}$ the solutions do not cut the h axis or they cut it once.

$m^2 \neq 0$ case

We are going to proceed now in the same manner as we did in the previous case. In the limit $t \rightarrow 0$, Eq.(6) with $m^2 \neq 0$ gives

$$\frac{1}{t^4} \left[n^2 - (1 + K)n^3 + Kn^4 \right] - \left(\frac{1}{2} + \frac{m^2}{t^{2n}} \right) \frac{1}{t^2} \left[n^2 - n \right] = 0. \quad (16)$$

From this equation we see that the term which comes from $\ddot{G}G^2$ in (6) can be neglected and the remaining approximate equation can be solved using the method described in [15]. In this case, we obtain

$$G = \left[e^{-\theta/2} \left(C_1 e^{\lambda\theta} + C_2 e^{-\lambda\theta} \right) \right]^{\frac{1}{K-1}}, \quad K \neq 1, \quad (17)$$

where

$$\lambda = \frac{[1 - 4\beta]^{1/2}}{2}, \quad \beta = (K - 1) \frac{m^2}{C^2}, \quad (18)$$

and $\theta = -C \int G^{-1} dt$.

$\beta \leq 1/4$: For $-\frac{1}{2} < K < 0$ and $1 < K$ the coefficient of the metric G diverges (vanishes) at a finite time $\theta = \theta_0$ when $\text{sgn } C_1 \neq \text{sgn } C_2$. In this case, we get

$$G \simeq at + b|t|^{1/K} \quad \text{for} \quad -\frac{1}{2} < K < 0 \quad \text{and} \quad 1 < K, \quad (19)$$

This two-parameter family of solutions behaves as $|t|^{1/K}$ when $t \rightarrow 0$. Also, G vanishes in the limit $\theta \rightarrow \pm\infty$ for $K < 1$ and its expansion is

$$G^\pm \simeq a^\pm t + b|t|^{2-K+\frac{m^2}{(a^\pm)^2}} \quad \text{for} \quad K < K_0 = 1 + \frac{m^2}{(a^\pm)^2}, \quad (20)$$

where $a^\pm = C [1/2 \pm \lambda] / (K - 1)$.

$\beta > 1/4$: Eq.(17) describes damped oscillatory solutions. These solutions are compatible only with an effective negative cosmological constant [9].

In the general case Eq.(7) can be written as the equation of motion for a dissipative or antidissipative system as follows

$$\frac{d}{d\eta} \left[\frac{h'^2}{2} + \mathcal{V}(h) \right] = 2m^2 e^{-2\eta} \ln h - \left\{ (1 + K) - \frac{1 + 2m^2 e^{-2\eta}}{2h^2} \right\} h'^2, \quad (21)$$

where now the “potential” is

$$\mathcal{V}(h) = \frac{Kh^2}{2} - \left\{ \frac{1}{2} + m^2 e^{-2\eta} \right\} \ln h, \quad (22)$$

and the local extreme points will be given by

$$h_0^2 = \frac{1 + 2m^2 e^{-2\eta}}{2K}. \quad (23)$$

Requiring that $K > 0$, the extreme points are actually minima and the right-hand side of Eq.(21) is negative definite, so that the bracket on the l.h.s of Eq.(21) defines a Liapunov function. Near the equilibrium points the approximate equation governing the trajectories on the phase plane, for $t \rightarrow \infty$, is given by Eq.(23) whose solution is

$$G_{min} = \pm \left\{ -\frac{m^2}{2} e^{\sqrt{\frac{1}{2K}}(t-t_0)} + e^{-\sqrt{\frac{1}{2K}}(t-t_0)} \right\}, \quad (24)$$

showing that the final anisotropic solution is asymptotically stable and coincides with the solution (8). For $k^2 > 2$ the late-time behaviour of an inhomogeneization of a Bianchi I model [10] is like that of Bianchi VI₀.

The limit behaviour of the scalar curvature when $t \rightarrow 0$ has two different possibilities depending on the values of K :

$$R \sim \begin{cases} t^{-[2+\frac{1}{K}]} & \text{for } -\frac{1}{2} < K < 0 \quad \text{and} \quad 1 < K, \\ t^{-[\frac{3}{2}+(\frac{m}{a^\pm}-\frac{k}{2})^2]} & \text{for } 0 < K < K_0. \end{cases} \quad (25)$$

We conclude that there exists a two-parameter family of singular solutions which describe a universe that begins from (or ends in) a singularity for any values of the constants m , C and k^2 . For these solutions the scalar curvature diverges when $t \rightarrow 0$. However, for $-\frac{1}{2} < K < 0$ it vanishes asymptotically and the solution remains regular.

We can also see that the shear-expansion ratio,

$$\frac{\sigma}{\Theta} = \frac{\sqrt{6}}{3} \frac{\dot{f}G - \dot{G}}{\dot{f}G + 2\dot{G}}, \quad (26)$$

which is usually considered as an indicator of the isotropization, does not vanish for the asymptotical solution given by (8). Therefore, we can say that none of the solutions with $k^2 > 2$, for which that study was valid, isotropize.

Some results of this paper are related to those ones obtained in [8], [9] and [10]. Solutions with $k^2 < 2$ show a power-law inflationary behaviour [16] while solutions with $k^2 > 2$ do not inflate or isotropize, since Bianchi VI models do not have FRW as particular solutions [3].

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